

LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

M.Sc. DEGREE EXAMINATION – MATHEMATICS

THIRD SEMESTER – NOVEMBER 2007

MT 3805 - ANALYTIC NUMBER THEORY

AB26

Date : 29/10/2007
Time : 9:00 - 12:00

Dept. No.

Max. : 100 Marks

ANSWER ALL QUESTIONS

I a) Write a product formula for $\varphi(n)$.

[OR]

b) Prove that if $n \geq 1$ then $\sum_{d|n} \Lambda(d) = \log n$. (5)

c) Prove that Euler's function $\varphi(n)$ has the following properties:

i) $\varphi(p^\alpha) = p^\alpha - p^{\alpha-1}$ for p and $\alpha \geq 1$.

ii) $\varphi(mn) = \varphi(m)\varphi(n)\varphi\left(\frac{d}{\varphi(d)}\right)$, where $d = (m, n)$.

iii) $\varphi(mn) = \varphi(m)\varphi(n)$ if $(m, n) = 1$.

iv) $\varphi(n)$ is even for $n \geq 3$. Moreover, if n has r distinct odd prime factors, then $2^r / \varphi(n)$.
(4 + 5 + 2 + 4)

[OR]

d) i) Prove that if $n \geq 1$ then $\sum_{d|n} \Lambda(d) = \log n$.

ii) Let the arithmetic function f be multiplicative. Then prove that f is completely multiplicative if, and only if $f^{-1}(n) = \mu(n)f(n)$ for all $n \geq 1$. (8 + 7)

II a) Write a note on the test for divisibility by 3.

[OR]

b) Derive the average order of $\varphi(n)$. (5)

c) State and prove Euler's summation formula. (15)

[OR]

d) If $x \geq 1$ prove that

i) $\sum_{n \leq x} \frac{1}{n} = \log x + C + O\left(\frac{1}{x}\right)$.

ii) $\sum_{n > x} \frac{1}{n^s} = O(x^{1-s})$. (8 + 7)

III a) Let $F_n = 2^{2^n} + 1$. Prove that F_5 is composite.

[OR]

b) State and prove Wilson's theorem. (5)

c) State and prove Chinese remainder theorem. Also write an application of the Chinese remainder theorem. (15)

[OR]

d) i) State and prove Lagrange's theorem for polynomial congruences mod p .

ii) Solve the congruence $25x \equiv 15 \pmod{120}$. (10 + 5)

IV a) Prove that Legendre's symbol is completely multiplicative.

[OR]

b) Find the value of $\left(\frac{2}{p}\right)$ where p is odd prime. (5)

c) State and prove Gauss' lemma.

[OR]

d) State and prove Euler's criterion. (15)

V a) Write an application of the reciprocity law.

[OR]

b) Derive Euler's recursion formula for $p(n)$. (5)

c) State and prove Euler's pentagonal-number theorem.

[OR]

d) State and prove Quadratic reciprocity law. (15)
